

# ON INJECTIVE DIMENSION OF LOCAL COHOMOLOGY OVER REGULAR RING

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**ABSTRACT.** Let  $R$  be a regular local ring containing a field  $k$  of characteristic zero and  $I$  be an ideal of  $R$ . In this paper we study injective dimension of local cohomology module  $H_I^i(R)$ . We prove if  $\dim_R(H_I^d(R)) \leq 3$  then  $\dim_R(H_I^d(R)) - 1 \leq \text{inj. dim}_R(H_I^d(R))$ . Also we show that if  $R = k[[x_1, \dots, x_n]]$  and  $\dim_R(H_I^d(R)) \leq 4$  then  $\dim_R(H_I^d(R)) - 1 \leq \text{inj. dim}_R(H_I^d(R))$ . For regular local ring containing an arbitrary field  $k$  we show that if  $\text{ht } I = d$  then  $\text{inj. dim}_R(H_I^d(R)) = \dim_R(H_I^d(R))$ .

## 1. INTRODUCTION

Throughout this paper,  $R$  is a commutative Noetherian ring with unit. If  $M$  is an  $R$ -module and  $I \subset R$  is an ideal, we denote the  $i$ -th local cohomology of  $M$  with support in  $I$  by  $H_I^i(M)$ .

In a remarkable paper, [5], Lyubeznik used  $\mathcal{D}$ -modules to prove if  $R$  is any regular ring containing a field of characteristic 0 and  $I$  is an ideal of  $R$ , then

- a)  $H_{\mathfrak{m}}^i(H_I^i(R))$  is injective for every maximal ideal  $\mathfrak{m}$  of  $R$ .
- b)  $\text{inj. dim}_R(H_I^i(R)) \leq \dim_R(H_I^i(R))$ .
- c) For every maximal ideal  $\mathfrak{m}$  of  $R$  the set of associated primes of  $H_I^i(R)$  contained in  $\mathfrak{m}$  is finite.
- d) All the bass numbers of  $H_I^i(R)$  are finite.

Here  $\text{inj. dim}_R(H_I^i(R))$  stands for the injective dimension of  $H_I^i(R)$ ,  $\dim_R(H_I^i(R))$  denotes the dimension of the support of  $H_I^i(R)$  in  $\text{Spec}(R)$  and the  $j$ -th Bass number of an  $R$ -module  $M$  with respect to a prime ideal  $\mathfrak{p}$  is defined as  $\mu_j(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})} \text{Ext}_{R_{\mathfrak{p}}}^j(k(\mathfrak{p}), M_{\mathfrak{p}})$  where  $k(\mathfrak{p})$  is the residue field of  $R_{\mathfrak{p}}$ .

By Lyubeznik results, the injective dimension of  $H_I^i(R)$  is bounded by its dimension. A question of Hellus [4] asks when  $\text{inj. dim}_R(H_I^i(R)) = \dim_R(H_I^i(R))$ . He proved the equality  $\text{inj. dim}_R(H_I^i(R)) = \dim_R(H_I^i(R))$  for regular local ring  $R$  which contains a field and cofinite local cohomology  $H_I^i(R)$ , see [4, Corollary 2.6]. On the other hand, he presented two counterexamples for this equality in which  $\text{inj. dim}_R(H_I^i(R)) = 0$  but  $\dim_R(H_I^i(R)) = 1$ , see [4, Example 2.9, 2.11]. Also for polynomial ring  $R = k[x_1, \dots, x_n]$  with field  $k$  of characteristic zero, Puthenpurakal, [7, Corollary 1.2], proved  $\text{inj. dim}_R(H_I^i(R)) = \dim_R(H_I^i(R))$ .

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In this paper, motivated by these results, we attempt to obtain results on the injective dimension of local cohomology module over regular ring. More explicitly we are interested in the following question:

*Question 1.1.* Let  $R$  be a regular local ring which contains a field  $k$  of characteristic zero. Is  $\dim_R(H_1^i(R)) - 1 \leq \text{inj. dim}_R(H_1^i(R))$ .

In this paper, we give a partial positive answer for Question 1.1. Namely:

**Theorem 1.2.** *Let  $(R, \mathfrak{m})$  be a regular local ring of dimension  $n$  containing a field of characteristic zero. Let  $I$  be an ideal of  $R$  and  $\dim_R(H_1^i(R)) \leq 3$ . Then  $\dim_R(H_1^i(R)) - 1 \leq \text{inj. dim}_R(H_1^i(R))$ .*

**Theorem 1.3.** *Let  $k$  be a field of characteristic zero and  $R = k[[x_1, \dots, x_n]]$ . Let  $I$  be an ideal of  $R$  such that  $\dim_R(H_1^i(R)) \leq 4$ . Then  $\dim_R(H_1^i(R)) - 1 \leq \text{inj. dim}_R(H_1^i(R))$ .*

Also we give following partial positive answer to the question of Hellus [4]. Let  $I$  be an ideal of a ring  $R$ . By  $\text{ht}_R(I)$ , we mean the height of the ideal  $I$ .

**Proposition 1.4.** *Let  $(R, \mathfrak{m})$  be a regular local ring of dimension  $n$  containing a field and  $I$  be an ideal of  $R$  such that  $\text{ht}_R(I) = d$ . Then  $\text{inj. dim}_R(H_1^d(R)) = \dim_R(H_1^d(R))$ .*

This manuscript is organized as follows. In section 2, we recall some definitions and properties of  $\mathcal{D}$ -modules. Later, in section 3, we discuss three lemmas which will help in proving Theorems 1.2 and 1.3. In section 4, we prove Theorems 1.2 and 1.3 and Proposition 1.4.

## 2. PRELIMINARIES

In this section we review the theory of  $\mathcal{D}$ -modules. The theory of  $\mathcal{D}$ -modules has become an indispensable tool in commutative algebra. Since its initial use to study local cohomology,  $\mathcal{D}$ -modules have been used to study tight closure,  $F$ -jumping coefficients, hyperplane arrangements, Nakai conjecture and in other commutative algebra contexts.

Let  $k$  be a field of characteristic 0 and let  $R$  denote the formal power series ring  $k[[x_1, \dots, x_n]]$  in  $n$  variables over  $k$ . Let  $\mathcal{D} = \mathcal{D}(R, k)$  denote the subring of the  $k$ -vector space endomorphisms of  $R$  generated by  $R$  and the usual differential operators  $\delta_1, \dots, \delta_n$ , defined formally, so that  $\delta_i f = \frac{\partial f}{\partial x_i}$ . We simply say  $\mathcal{D}$ -modules for left  $\mathcal{D}(R, k)$  modules.  $\mathcal{D}(R, k)$  is left and right Noetherian [1, Lemma 3.1.6]. This implies that every finitely generated  $\mathcal{D}$ -modules is Noetherian. The natural action of  $\mathcal{D}(R, k)$  on  $R$  makes  $R$  as a  $\mathcal{D}$ -module. In addition if  $M$  is a  $\mathcal{D}$ -module and  $S \subset R$  is a multiplicative system of elements, using the quotient rule,  $M_S$  carries a natural structure of  $\mathcal{D}$ -module. Let  $I$  be an ideal of  $R$ . The Čech complex on a generating set for  $I$  is a complex of  $\mathcal{D}$ -modules; it then follows that each local cohomology module  $H_1^i(R)$  is a  $\mathcal{D}$ -module. There exists a remarkable class of finitely generated  $\mathcal{D}$ -modules, called holonomic  $\mathcal{D}$ -modules. See [1, Definition 7.12] for a definition of a holonomic  $\mathcal{D}$ -module.

*Remark 2.1.* Some of the properties of holonomic modules are as follows:

- a)  $R$  with its natural structure of  $\mathcal{D}(R, k)$ -module is holonomic [1, Theorem 3.3.2].
- b) If  $M$  is holonomic and  $f \in R$ , then  $M_f$  is holonomic [1, Theorem 3.4.1].
- c) Let  $M$  be a holonomic  $\mathcal{D}$ -module. Assume  $\text{Ass}_R M = \{\mathfrak{p}\}$  and  $M$  is  $\mathfrak{p}$ -torsion. Then there exists  $h \in R \setminus \mathfrak{p}$  such that  $\text{Hom}_R(R/\mathfrak{p}, M)_h$  is finitely generated as a  $R_h$ -module [7, Proposition 2.3].
- d) The holonomic modules form an abelian subcategory of the category of the category of  $\mathcal{D}$ -modules, which is closed under formation of submodules, quotient modules and extensions [5, 2.2c]. So  $H_1^i(R)$  is a holonomic  $\mathcal{D}$ -module.

We will use the following several times in this paper.

*Remark 2.2.* Adopt the above notations.

- a) Let  $\mathfrak{p}$  be a prime ideal of  $R$  and let  $E_R(R/\mathfrak{p})$  denote the injective envelope of  $R/\mathfrak{p}$ . Assume  $\text{ht}_R(\mathfrak{p}) = d$ . Recall that  $E_R(R/\mathfrak{p}) = H_{\mathfrak{p}}^d(R)_{\mathfrak{p}}$ . It follows that  $E_R(R/\mathfrak{p})$  is a  $\mathcal{D}$ -module and the natural inclusion  $H_{\mathfrak{p}}^d(R) \rightarrow E_R(R/\mathfrak{p})$  is  $\mathcal{D}(R, k)$ -linear. It implies that if  $M$  is a  $\mathcal{D}$ -module, then the minimal injective resolution of  $M$  in  $R$  is a complex of  $\mathcal{D}$ -modules and  $\mathcal{D}$ -linear maps.
- b) Let  $(S, \mathfrak{m})$  be the regular local ring which contains a field of characteristic zero. We denote by  $\hat{S}$  the completion of  $S$  with respect to the maximal ideal  $\mathfrak{m}$ . By Cohen structure theorem  $\hat{S} = k[[x_1, \dots, x_n]]$  where  $k$  is a field of characteristic zero. Let  $\mathfrak{p}$  be the prime ideal of  $S$  such that  $\text{ht}_S(\mathfrak{p}) = d$ . Recall that  $E_S(S/\mathfrak{p}) = H_{\mathfrak{p}}^d(S)_{\mathfrak{p}}$ . Then  $E_S(S/\mathfrak{p}) \otimes_S \hat{S} \cong H_{\mathfrak{p}\hat{S}}^d(\hat{S})_{\mathfrak{p}}$ , see [3, Theorem 4.3.2]. Hence  $E_S(S/\mathfrak{p}) \otimes_S \hat{S}$  has a structure of  $\mathcal{D}(\hat{S}, k)$ -module. We simply say that  $E_S(S/\mathfrak{p}) \otimes_S \hat{S}$  is a  $\mathcal{D}$ -module.
- c) Adopt the notations of (b). Let  $I$  be an ideal of  $S$  and let

$$0 \longrightarrow H_I^i(S) \xrightarrow{\eta} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \dots \xrightarrow{d^{t-1}} E^t \xrightarrow{d^t} 0$$

be the minimal injective resolution for  $H_I^i(S)$ . Keep in mind that  $\hat{S}$  is faithfully flat  $S$ -module. It follows that we have the following exact sequence of  $\hat{S}$ -modules where the maps are the natural :

$$0 \longrightarrow H_I^i(S) \otimes_S \hat{S} \xrightarrow{\hat{\eta}} E^0 \otimes_S \hat{S} \xrightarrow{\hat{d}^0} E^1 \otimes_S \hat{S} \xrightarrow{\hat{d}^1} \dots \xrightarrow{\hat{d}^{t-1}} E^t \otimes_S \hat{S} \xrightarrow{\hat{d}^t} 0.$$

We want to show that this sequence is a sequence of  $\mathcal{D}$ -modules and  $\mathcal{D}$ -linear maps. Note that with respect to (b) all of the terms of the latter sequence are  $\mathcal{D}$ -modules. So we only need to show that  $\hat{\eta}$  and  $\hat{d}^i$ 's are  $\mathcal{D}$ -linear. We prove it by induction. Clearly  $\hat{\eta}$  is  $\mathcal{D}$ -linear. Set  $\hat{d}^{-1} := \hat{\eta}$ . Assume that  $\hat{d}^r$ 's are  $\mathcal{D}$ -linear for  $r < j$ . So  $\text{im}(\hat{d}^{j-1})$  is  $\mathcal{D}$ -module and therefore  $\frac{E^j}{\text{im}(\hat{d}^{j-1})} \otimes_S \hat{S} \cong \frac{E^j \otimes_S \hat{S}}{\text{im}(\hat{d}^{j-1})}$  is  $\mathcal{D}$ -module. There is natural inclusion  $\phi : \frac{E^j}{\text{im}(\hat{d}^{j-1})} \otimes_S \hat{S} \rightarrow E^{j+1} \otimes_S \hat{S}$  which is  $\mathcal{D}$ -linear. Since  $\hat{d}^j = \phi\pi$  where  $\pi$  is the natural surjection  $E^j \otimes_S \hat{S} \rightarrow \frac{E^j \otimes_S \hat{S}}{\text{im}(\hat{d}^{j-1})}$ , we see  $\hat{d}^j$  is  $\mathcal{D}$ -linear.

## 3. PRELIMINARY LEMMAS

In this section, we establish three lemmas which will enable us to prove theorems in the next section. Let  $(R, \mathfrak{m})$  be a local ring and  $M$  be an  $R$ -module. By  $\text{depth}_R(M)$ , we mean the length of the maximal  $M$ -regular sequence in  $\mathfrak{m}$ .

**Lemma 3.1.** *Let  $k$  be a field of characteristic zero and  $R = k[[x_1, \dots, x_n]]$ . Let  $\mathfrak{p}$  be a prime ideal of  $R$  of height less than  $n - 1$ . Then  $E_R(R/\mathfrak{p})$  is not a holonomic  $\mathcal{D}$ -module.*

*Proof.* Suppose on the contrary  $E_R(R/\mathfrak{p})$  is a holonomic  $\mathcal{D}$ -module. It is well-known that  $\Gamma_{\mathfrak{p}}(E_R(R/\mathfrak{p})) = E_R(R/\mathfrak{p})$  and  $\text{Ass}_R E(R/\mathfrak{p}) = \mathfrak{p}$ . Then by Remark 2.1(c), there exists  $h \in R \setminus \mathfrak{p}$  such that  $\text{Hom}_R(R/\mathfrak{p}, E_R(R/\mathfrak{p}))_h$  is a finitely generated  $R_h$  module. Pick  $\mathfrak{q} \in \text{Spec}(R)$  which contains  $\mathfrak{p}$  such that  $\text{ht}_R(\mathfrak{q}) = n - 1$  and  $h \notin \mathfrak{q}$ . It follows that  $M := \text{Hom}_R(R/\mathfrak{p}, E_R(R/\mathfrak{p}))_{\mathfrak{q}}$  is a non-zero finitely generated  $R_{\mathfrak{q}}$  module. On the other hand  $M$  is an injective  $R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}$  module and therefore  $\text{depth}_{R_{\mathfrak{q}}}(R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}) = \text{inj. dim}_{R_{\mathfrak{q}}}(M) = 0$ . This contradicts with the fact that  $R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}$  is a domain of dimension greater than one.  $\square$

Let  $I$  be an ideal of a ring  $R$ . By  $\min_R(I)$ , we mean the set of all minimal prime ideals of  $I$ .

**Lemma 3.2.** *Let  $(R, \mathfrak{m})$  be a regular local ring of dimension  $n$  which contains a field of characteristic zero. Assume  $P \in \text{Spec}(R)$  such that  $\text{ht}_R(P) = n - 2$ . Let  $\hat{R}$  denote the completion of  $R$  with respect to the maximal ideal  $\mathfrak{m}$ . Then  $E_R(R/P) \otimes_R \hat{R}$  is non-holonomic  $\mathcal{D}$ -module.*

*Proof.* Recall that  $E_R(R/P) \cong H_P^{n-2}(R)_P$  and  $E_R(R/P) \otimes_R \hat{R} \cong H_P^{n-2}(R)_P \otimes_R \hat{R} \cong H_{P\hat{R}}^{n-2}(\hat{R})_P$ . In view of Remark 2.2(b),  $E_R(R/P) \otimes_R \hat{R}$  has a structure of  $\mathcal{D}(\hat{R}, k)$ -module where  $k$  is a field of characteristic zero which contained in  $\hat{R}$ . We simply say  $E_R(R/P) \otimes_R \hat{R}$  is a  $\mathcal{D}$ -module. Let  $\min_{\hat{R}}(P\hat{R}) = \{\mathfrak{q}_1, \dots, \mathfrak{q}_s\}$ . If  $\min_{\hat{R}}(P\hat{R})$  has one element we are done by Lemma 3.1. Hence we assume  $s > 1$ . There are infinitely many prime  $\mathfrak{p} \in \text{Spec}(R)$  such that  $\text{ht}_R(\mathfrak{p}) = n - 1$  and  $P \subsetneq \mathfrak{p}$ , see [8, Theorem 31.2]. For such  $\mathfrak{p}$ ,  $\text{ht}_{\hat{R}}(\mathfrak{p}\hat{R}) = n - 1$  and  $\mathfrak{p}\hat{R} \cap R = \mathfrak{p}$ . Thus we can assume that  $\text{ht}_{\hat{R}}(\mathfrak{q}_1) = n - 2$  and there are infinitely many prime  $\mathfrak{q} \in \text{Spec}(\hat{R})$  which contains  $\mathfrak{q}_1$  and  $P \subsetneq \mathfrak{q} \cap R$ .

Suppose on the contrary that  $H_{P\hat{R}}^{n-2}(\hat{R})_P$  is holonomic. We have the Mayer-Vietoris sequence

$$0 \rightarrow H_{\mathfrak{q}_1}^{n-2}(\hat{R}) \oplus H_{\mathfrak{q}_2 \cap \dots \cap \mathfrak{q}_s}^{n-2}(\hat{R}) \rightarrow H_{P\hat{R}}^{n-2}(\hat{R}).$$

Hence  $H_{\mathfrak{q}_1}^{n-2}(\hat{R})_P$  is a holonomic  $\mathcal{D}$ -module. It is clear that  $\text{Ass}_{\hat{R}}(H_{\mathfrak{q}_1}^{n-2}(\hat{R})_P) = \mathfrak{q}_1$ . Indeed let  $m/s \in H_{\mathfrak{q}_1}^{n-2}(\hat{R})_P$  such that  $m \in H_{\mathfrak{q}_1}^{n-2}(\hat{R})$  and  $s \in R \setminus P$ . If  $r \in \hat{R}$  such that  $r.m/s = 0$ , then there exists  $r' \in R \setminus P \subseteq \hat{R} \setminus \mathfrak{q}_1$  such that  $r'.rm = 0$ . Keep in mind that  $\text{Ass}_{\hat{R}}(H_{\mathfrak{q}_1}^{n-2}(\hat{R})) = \mathfrak{q}_1$ . So  $r'.r \in \mathfrak{q}_1$  and thus  $r \in \mathfrak{q}_1$ . Also  $\Gamma_{\mathfrak{q}_1}(H_{\mathfrak{q}_1}^{n-2}(\hat{R})_P) = H_{\mathfrak{q}_1}^{n-2}(\hat{R})_P$ . Then by Remark 2.1(c), there exists  $h \in \hat{R} \setminus \mathfrak{q}_1$  such that  $\text{Hom}_{\hat{R}}(\frac{\hat{R}}{\mathfrak{q}_1 \hat{R}}, H_{\mathfrak{q}_1}^{n-2}(\hat{R})_P)_h$  is finitely generated  $\hat{R}_h$  module. Since  $\mathfrak{q}_i \not\subseteq \mathfrak{q}_1$  for all  $2 \leq i \leq s$ , we can pick  $t_i \in \mathfrak{q}_i \setminus \mathfrak{q}_1$  for all  $2 \leq i \leq s$ . Thus  $t = t_2 \dots t_s h \notin \mathfrak{q}_1$ . Note that the set of minimal prime ideals of

the ideal generated by  $t$  and  $\mathfrak{q}_1$  is finite. Then by assumption on choosing  $\mathfrak{q}_1$ , we can pick  $\mathfrak{q} \in \text{Spec}(\hat{R})$  which contains  $\mathfrak{q}_1$  and  $t \notin \mathfrak{q}$  such that  $\text{ht}_R(\mathfrak{q} \cap R) = n - 1$ .

Thus  $\text{Hom}_{\hat{R}_{\mathfrak{q}}}(\frac{\hat{R}_{\mathfrak{q}}}{\mathfrak{q}_1 \hat{R}_{\mathfrak{q}}}, (H_{\mathfrak{q}_1}^{n-2}(\hat{R})_P)_{\mathfrak{q}})$  is a finitely generated  $\hat{R}_{\mathfrak{q}}$ -module. Since  $\min_{\hat{R}_{\mathfrak{q}}}(P \hat{R}_{\mathfrak{q}}) = \mathfrak{q}_1 \hat{R}_{\mathfrak{q}}$ , then  $H_{\mathfrak{q}_1 \hat{R}_{\mathfrak{q}}}^{n-2}(\hat{R}_{\mathfrak{q}}) = H_{P \hat{R}_{\mathfrak{q}}}^{n-2}(\hat{R}_{\mathfrak{q}})$ . Also  $\frac{\hat{R}_{\mathfrak{q}}}{P \hat{R}_{\mathfrak{q}}}$  has a filtration of  $\hat{R}_{\mathfrak{q}}$ -modules such that quotients of it are isomorph to  $\frac{\hat{R}_{\mathfrak{q}}}{\mathfrak{q}_1 \hat{R}_{\mathfrak{q}}}$  or  $\frac{\hat{R}_{\mathfrak{q}}}{\mathfrak{q} \hat{R}_{\mathfrak{q}}}$ , as  $\hat{R}_{\mathfrak{q}}$ -module. Thus  $\text{Hom}_{\hat{R}_{\mathfrak{q}}}(\frac{\hat{R}_{\mathfrak{q}}}{P \hat{R}_{\mathfrak{q}}}, (H_{P \hat{R}}^{n-2} \hat{R}_P)_{\mathfrak{q}})$  is a finitely generated  $\hat{R}_{\mathfrak{q}}$ -module.

Look at the faithful map  $R_{\mathfrak{q} \cap R} \rightarrow \hat{R}_{\mathfrak{q}}$ . We have following isomorphisms:

$$\begin{aligned} & \text{Hom}_{R_{\mathfrak{q} \cap R}}(\frac{R_{\mathfrak{q} \cap R}}{P R_{\mathfrak{q} \cap R}}, (H_P^{n-2}(R)_P)_{\mathfrak{q} \cap R} \otimes_{R_{\mathfrak{q} \cap R}} \hat{R}_{\mathfrak{q}}) \cong \text{Hom}_{\hat{R}_{\mathfrak{q}}}(\frac{R_{\mathfrak{q} \cap R}}{P R_{\mathfrak{q} \cap R}} \otimes_{R_{\mathfrak{q} \cap R}} \hat{R}_{\mathfrak{q}}, (H_P^{n-2}(R)_P)_{\mathfrak{q} \cap R} \otimes_{R_{\mathfrak{q} \cap R}} \hat{R}_{\mathfrak{q}}) \\ & \cong \text{Hom}_{\hat{R}_{\mathfrak{q}}}((R/P \otimes_R R_{\mathfrak{q} \cap R}) \otimes_{R_{\mathfrak{q} \cap R}} \hat{R}_{\mathfrak{q}}, ((H_P^{n-2}(R) \otimes_R R_P) \otimes_{R_{\mathfrak{q} \cap R}} \hat{R}_{\mathfrak{q}})) \\ & \cong \text{Hom}_{\hat{R}_{\mathfrak{q}}}(R/P \otimes_R \hat{R}_{\mathfrak{q}}, (H_P^{n-2}(R) \otimes_R R_P) \otimes_R \hat{R}_{\mathfrak{q}}) \cong \\ & \text{Hom}_{\hat{R}_{\mathfrak{q}}}(R/P \otimes_R (\hat{R} \otimes_{\hat{R}} \hat{R}_{\mathfrak{q}}), (H_P^{n-2}(R) \otimes_R R_P) \otimes_R (\hat{R} \otimes_{\hat{R}} \hat{R}_{\mathfrak{q}})) \cong \text{Hom}_{\hat{R}_{\mathfrak{q}}}(\frac{\hat{R}_{\mathfrak{q}}}{P \hat{R}_{\mathfrak{q}}}, (H_{P \hat{R}}^{n-2}(\hat{R})_P)_{\mathfrak{q}}) \end{aligned}$$

Therefore  $\text{Hom}_{R_{\mathfrak{q} \cap R}}(\frac{R_{\mathfrak{q} \cap R}}{P R_{\mathfrak{q} \cap R}}, E_R(R/P)_{\mathfrak{q} \cap R}) \cong \text{Hom}_{R_{\mathfrak{q} \cap R}}(\frac{R_{\mathfrak{q} \cap R}}{P R_{\mathfrak{q} \cap R}}, (H_P^{n-2}(R)_P)_{\mathfrak{q} \cap R})$  is a non-zero finitely generated  $R_{\mathfrak{q} \cap R}$ -module. Again, it is contradiction because  $\frac{R_{\mathfrak{q} \cap R}}{P R_{\mathfrak{q} \cap R}}$  is a domain of dimension greater than one.  $\square$

*Example 3.3.* Let  $k$  be a field of characteristic zero and  $R = k[[x_1, \dots, x_5]]$ .

Let  $I := (x_1, x_2)R \cap (x_3, x_4)R \cap (x_5, x_1)R$  and  $\mathfrak{m}$  be the maximal ideal of  $R$ . Then there exists following minimal injective resolution for  $H_I^3(R)$ , see [4, Examples 2.11],

$$0 \rightarrow H_I^3(R) \rightarrow E_R(R/(x_1, x_2, x_3, x_4)R) \oplus E_R(R/(x_1, x_3, x_4, x_5)R) \rightarrow E_R(R/\mathfrak{m}) \rightarrow 0.$$

In view of Remark 2.2(a), this exact sequence is a sequence of  $\mathcal{D}$ -modules and  $\mathcal{D}$ -homomorphisms. Since  $E_R(R/\mathfrak{m}) \cong H_{\mathfrak{m}}^5(R)$  then  $H_I^3(R)$  and  $H_{\mathfrak{m}}^5(R)$  are holonomic  $\mathcal{D}$ -modules. Putting this along with Remark 2.1(d)  $E_R(R/(x_1, x_2, x_3, x_4)R)$  and  $E_R(R/(x_1, x_3, x_4, x_5)R)$  are holonomic. So Lemma 3.1 is not true for prime ideal  $\mathfrak{q}$  of height  $n - 1$  or  $n$ . Indeed we show in the next section that  $E_R(R/\mathfrak{q})$  is holonomic for prime ideal  $\mathfrak{q}$  of height  $n - 1$  or  $n$ .

Let  $M$  be a finitely generated module over a Cohen-Macaulay ring  $R$  such that  $\text{inj. dim}_R(M)$  is finite and therefore it equals to  $\dim R$ . Then it is elementary to prove that if  $\mu_{\dim R}(\mathfrak{p}, M) > 0$  then  $\mathfrak{p}$  is a maximal ideal in  $R$ , use [2, Proposition 3.1.13]. Although this fact is not true for  $R$ -module  $M$  that is not finitely generated. For example Let  $\mathfrak{p}$  be a prime ideal of  $R$  and  $M$  be the injective envelope of  $R/\mathfrak{p}$ .

For polynomial ring  $R = k[x_1, \dots, x_n]$  with field  $k$  of characteristic zero, Puthenpurakal proved if  $\text{inj. dim}_R(H_I^i(R)) = c$  and  $\mu_c(\mathfrak{p}, H_I^i(R)) > 0$  for prime ideal  $\mathfrak{p}$  of  $R$ , then  $\mathfrak{p}$  is a maximal ideal of  $R$ , see [7, Theorem 1.1]. In the following lemma we generalize his theorem to the case  $R$  is a formal power series ring over a field  $k$  of characteristic zero.

**Lemma 3.4.** *Let  $k$  be a field of characteristic zero and  $R = k[[x_1, \dots, x_n]]$ . Let  $I$  be an ideal of  $R$  and  $\text{inj. dim}(H_I^i(R)) = c$ . Assume  $\mu_c(P, H_I^i(R)) \neq 0$  for prime ideal  $\mathfrak{p}$  of  $R$ . Then  $\text{ht}_R(\mathfrak{p}) \geq n - 1$ .*

*Proof.* By [5, Theorem 3.4(b)]  $(H_{\mathfrak{p}}^c(H_1^i(R)))_{\mathfrak{p}}$  is an injective  $R_{\mathfrak{p}}$ -module. Then  $(H_{\mathfrak{p}}^c(H_1^i(R)))_{\mathfrak{p}} \cong E_R(R/\mathfrak{p})^s$  where  $s$  is a positive integer, see [5, Theorem 3.4(d)]. So  $(H_{\mathfrak{p}}^c(H_1^i(R)))_{\mathfrak{p}}$  is an injective  $R$ -module. Therefore  $\mu_c(\mathfrak{p}, H_1^i(R)) = \mu_0(\mathfrak{p}, H_{\mathfrak{p}}^c(H_1^i(R))) > 0$ , see [5, Lemma 1.4]. Suppose on the contrary  $\text{ht}_R(\mathfrak{p}) \leq n - 2$ . Let  $\text{Ass}_R(H_{\mathfrak{p}}^c(H_1^i(R))) = \{\mathfrak{p}, \mathfrak{q}_1, \dots, \mathfrak{q}_m\}$ . Look at exact sequence:

$$0 \rightarrow \Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H_{\mathfrak{p}}^c(H_1^i(R))) \rightarrow H_{\mathfrak{p}}^c(H_1^i(R)) \rightarrow H_{\mathfrak{p}}^c(H_1^i(R))/\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H_{\mathfrak{p}}^c(H_1^i(R))) \rightarrow 0.$$

Since  $\mathfrak{p} \subsetneq \mathfrak{q}_i$ , we have  $P \notin \text{Ass}_R \Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H_{\mathfrak{p}}^c(H_1^i(R)))$ . Keep in mind that

$$\text{Ass}_R H_{\mathfrak{p}}^c(H_1^i(R)) = \text{Ass}_R \Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H_{\mathfrak{p}}^c(H_1^i(R))) \cup \text{Ass}_R H_{\mathfrak{p}}^c(H_1^i(R))/\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H_{\mathfrak{p}}^c(H_1^i(R))).$$

It follows that  $\text{Ass}_R H_{\mathfrak{p}}^c(H_1^i(R))/\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H_{\mathfrak{p}}^c(H_1^i(R))) = \{\mathfrak{p}\}$ .

Let  $f \in R \setminus \mathfrak{p}$ . Then the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H_{\mathfrak{p}}^c(H_1^i(R))) & \longrightarrow & H_{\mathfrak{p}}^c(H_1^i(R)) & \longrightarrow & H_{\mathfrak{p}}^c(H_1^i(R))/\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H_{\mathfrak{p}}^c(H_1^i(R))) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \eta \\ 0 & \longrightarrow & \Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H_{\mathfrak{p}}^c(H_1^i(R))_f) & \longrightarrow & H_{\mathfrak{p}}^c(H_1^i(R))_f & \longrightarrow & (H_{\mathfrak{p}}^c(H_1^i(R))/\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H_{\mathfrak{p}}^c(H_1^i(R))))_f \longrightarrow 0. \end{array}$$

Recall that  $\text{inj. dim } H_1^i(R) = c$ . Thus, there is an exact sequence

$$H_{(\mathfrak{p}, f)}^c(H_1^i(R)) \rightarrow H_{\mathfrak{p}}^c(H_1^i(R)) \rightarrow H_{\mathfrak{p}}^c(H_1^i(R))_f \rightarrow H_{(\mathfrak{p}, f)}^{c+1}(H_1^i(R)) = 0.$$

Hence, the natural map  $\eta$  is surjective. As  $f \notin \mathfrak{p}$ , we get that  $\eta$  is also injective. Thus,  $H_{\mathfrak{p}}^c(H_1^i(R))/\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H_{\mathfrak{p}}^c(H_1^i(R))) = (H_{\mathfrak{p}}^c(H_1^i(R))/\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H_{\mathfrak{p}}^c(H_1^i(R))))_f$  for all  $f \in R \setminus \mathfrak{p}$ . It follows that  $H_{\mathfrak{p}}^c(H_1^i(R))/\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H_{\mathfrak{p}}^c(H_1^i(R))) = (H_{\mathfrak{p}}^c(H_1^i(R))/\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H_{\mathfrak{p}}^c(H_1^i(R))))_{\mathfrak{p}}$ .

Note that  $(\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H_{\mathfrak{p}}^c(H_1^i(R))))_{\mathfrak{p}} = 0$ . We deduce that

$$H_{\mathfrak{p}}^c(H_1^i(R))_{\mathfrak{p}} \cong (H_{\mathfrak{p}}^c(H_1^i(R))/\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H_{\mathfrak{p}}^c(H_1^i(R))))_{\mathfrak{p}} \cong H_{\mathfrak{p}}^c(H_1^i(R))/\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H_{\mathfrak{p}}^c(H_1^i(R))).$$

Clearly  $H_{\mathfrak{p}}^c(H_1^i(R))/\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H_{\mathfrak{p}}^c(H_1^i(R)))$  is holonomic. Putting this along with  $(H_{\mathfrak{p}}^c(H_1^i(R)))_{\mathfrak{p}} \cong E_R(R/\mathfrak{p})^s$ , we conclude that  $E_R(R/\mathfrak{p})$  is holonomic. So we reach to a contradiction because by Lemma 3.1  $E_R(R/\mathfrak{p})$  can not be holonomic.  $\square$

*Remark 3.5.* a) There are examples for power series ring  $R$  over a field  $k$  of characteristic zero such that  $\dim_R H_1^i(R) = 1$  and  $\text{inj. dim}_R(H_1^i(R)) = 0$ , see for example [4, Examples 2.9, 2.11]. For these examples, there exists  $\mathfrak{p} \in \text{Ass}_R(H_1^i(R))$  such that  $\text{ht}_R(\mathfrak{p}) = n - 1$ . It is well-known that for all  $R$ -module  $M$ ,  $\mathfrak{q} \in \text{Ass}_R(M)$  if and only if  $\mu_0(\mathfrak{q}, M) > 0$ . It follows that  $\mu_0(\mathfrak{p}, H_1^i(R)) > 0$ .

b) Let  $k$  be a field of characteristic zero and  $R = k[[x_1, \dots, x_n]]$ . Let  $I$  be an ideal of  $R$ . Assume  $\dim(H_1^i(R)) = 2$ . There exists  $\mathfrak{q} \in \text{Ass}_R(H_1^i(R))$  such that  $\text{ht}_R(\mathfrak{q}) = n - 2$  and  $\mu_0(\mathfrak{q}, H_1^i(R)) > 0$ . Then  $1 \leq \text{inj. dim}_R(H_1^i(R))$  by Lemma 3.4. Thus we can say if  $\dim(H_1^i(R)) \leq 2$  then  $\dim(H_1^i(R)) - 1 \leq \text{inj. dim}(H_1^i(R))$ .

## 4. APPLICATION: INJECTIVE DIMENSION OF LOCAL COHOMOLOGY

In this section we prove our main results about injective dimension of local cohomology. First we prove two elementary lemmas.

**Lemma 4.1.** *Let  $R$  be a regular local ring which contains a field and  $I$  be an ideal of  $R$ . Let  $\text{inj. dim}_R(H_I^i(R)) = \dim_R(H_I^i(R)) = c$ . If  $\mu_c(\mathfrak{p}, H_I^i(R)) \neq 0$  for  $\mathfrak{p} \in \text{Spec}(R)$  then  $\mathfrak{p}$  is a maximal ideal of  $R$ .*

*Proof.* Let  $\dim(R) = n$ . We suppose on the contrary  $\text{ht}_R \mathfrak{p} \leq n-1$ . Thus  $\dim_{R_{\mathfrak{p}}}(H_I^i(R))_{\mathfrak{p}} \leq c-1$ . Since  $\mu_c(\mathfrak{p}, H_I^i(R)) \neq 0$ , we deduce that  $\text{inj. dim}_{R_{\mathfrak{p}}}(H_I^i(R))_{\mathfrak{p}} = c$ . But this is impossible because in view of [5, Theorem 3.4(b)] and [6, Theorem 1.4], we must have  $\text{inj. dim}_{R_{\mathfrak{p}}}(H_I^i(R))_{\mathfrak{p}} \leq \dim_{R_{\mathfrak{p}}}(H_I^i(R))_{\mathfrak{p}}$ .  $\square$

**Lemma 4.2.** *Let  $(R, \mathfrak{m})$  be a local ring of dimension  $n$ . Let  $\hat{R}$  denote the completion of  $R$  with respect to the maximal ideal  $\mathfrak{m}$ . Let  $M$  be an  $R$ -module. Then  $\dim_R(M) = \dim_{\hat{R}}(M \otimes_R \hat{R})$ .*

*Proof.* Let  $\dim_R(M) = d$ . There exists  $\mathfrak{p} \in \text{Supp}_R(M)$  such that  $d = \dim R/\mathfrak{p} = \dim \hat{R}/\mathfrak{p}\hat{R}$ . Thus there exists  $\mathfrak{q} \in \text{Spec}(\hat{R})$  such that  $\mathfrak{q}$  is minimal over  $\mathfrak{p}\hat{R}$  and  $\dim \hat{R}/\mathfrak{q}\hat{R} = d$ . We show that  $\mathfrak{q} \in \text{Supp}_{\hat{R}}(M \otimes_R \hat{R})$  and so  $\dim_{\hat{R}}(M \otimes_R \hat{R}) \geq d$ . It is clear that  $\mathfrak{q} \cap R = \mathfrak{p}$ . Hence the natural map  $R_{\mathfrak{p}} \rightarrow \hat{R}_{\mathfrak{q}}$  is faithfully flat. Thus

$$(M \otimes_R \hat{R}) \otimes_{\hat{R}} \hat{R}_{\mathfrak{q}} \cong M \otimes_R \hat{R}_{\mathfrak{q}} \cong M \otimes_R (R_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \hat{R}_{\mathfrak{q}}) \cong (M \otimes_R R_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} \hat{R}_{\mathfrak{q}}.$$

So  $M_{\mathfrak{q}} \neq 0$  as desired.

On the other hand let  $\dim_{\hat{R}}(M \otimes_R \hat{R}) = c$ . Thus there exists  $\mathfrak{q} \in \text{Supp}_{\hat{R}}(M \otimes_R \hat{R})$  such that  $\dim \hat{R}/\mathfrak{q}\hat{R} = c$ . Let  $\mathfrak{q} \cap R = \mathfrak{p}$ . Thus  $\dim R/\mathfrak{p} = \dim \hat{R}/\mathfrak{p}\hat{R} \geq \dim \hat{R}/\mathfrak{q}\hat{R} = c$ . So we only need to show that  $\mathfrak{p} \in \text{Supp}(M)$ . It is obvious by the isomorphism  $(M \otimes_R \hat{R})_{\mathfrak{q}} \cong (M \otimes_R R_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} \hat{R}_{\mathfrak{q}}$ .  $\square$

We now give

Proof of Proposition 1.4. Assume  $\text{ht}_R(I) = d$ . Let  $\min_R(I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_s\} \cup \{\mathfrak{q}_1, \dots, \mathfrak{q}_t\}$  such that  $\text{ht}_R(\mathfrak{p}_i) = d$  and  $\text{ht}_R(\mathfrak{q}_i) > d$ . Set  $I' := \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_s$  and  $I'' = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_t$ . We have the Mayer-vietoris sequence

$$H_{I'+I''}^d(R) \rightarrow H_{I'}^d(R) \oplus H_{I''}^d(R) \rightarrow H_I^d(R) \rightarrow H_{I'+I''}^{d+1}(R).$$

Since  $H_{I'+I''}^d(R) = H_{I'+I''}^{d+1}(R) = H_{I''}^d(R) = 0$  we deduce that  $H_I^d(R) \cong H_{I'}^d(R)$ . Thus without loss of generality, we can assume that all minimal prime ideals of  $I$  have height  $d$ .

There exists the spectral sequence  $H_{\mathfrak{m}}^i(H_I^j(R)) \Rightarrow H_{\mathfrak{m}}^{i+j}(R)$ . By using Hartshorne-Lichtenbaum theorem, we easily see that  $\text{inj. dim}(H_I^i(R)) \leq \dim(H_I^i(R)) \leq n - (i+1)$  for all  $i > d$ . So on the line  $y+x=n$  of the spectral sequence  $H_{\mathfrak{m}}^i(H_I^j(R)) \Rightarrow H_{\mathfrak{m}}^{i+j}(R)$ , we have  $(H_{\mathfrak{m}}^{n-i}(H_I^i(R))) = 0$  for all  $i > d$ . By definition of the spectral sequence  $H_{\mathfrak{m}}^i(H_I^j(R)) \Rightarrow H_{\mathfrak{m}}^{i+j}(R)$  there exists a filtration

$$0 \subseteq \dots \subseteq F^t H_n \subseteq F^{t-1} H_n \subseteq \dots \subseteq F^s H_n = H_{\mathfrak{m}}^n(R)$$



of  $H_{\mathfrak{m}}^n(R)$  such that  $E_{\infty}^{i,n-i} \cong \frac{F^i H_n}{F^{i+1} H_n}$ . Since  $E_{\infty}^{n-d-i,d+i} = 0$  for all  $i \geq 1$  then  $E_{\infty}^{n-d,d} \cong H_{\mathfrak{m}}^n(R)$ . Note that  $E_{\infty}^{n-d,d}$  is the quotient of  $H_{\mathfrak{m}}^{n-d}(H_1^d(R))$ . Then  $H_{\mathfrak{m}}^{n-d}(H_1^d(R))$  must be non-zero. It implies that  $\dim_R(H_1^d(R)) = n - d \leq \text{inj. dim}_R(H_1^d(R))$ .

We also need two following lemmas to prove Theorems 1.2 and 1.3.

**Lemma 4.3.** *Let  $k$  be a field of characteristic zero and  $R = k[[x_1, \dots, x_n]]$ . Let  $\mathfrak{p} \in \text{Spec}(R)$  such that  $\text{ht}_R(\mathfrak{p}) = n - 1$ . Then  $E_R(R/\mathfrak{p})$  is a holonomic  $\mathcal{D}$ -module.*

*Proof.* Recall that  $E_R(R/\mathfrak{p})$  has a structure of  $\mathcal{D}$ -module, see Remark 2.2 (a). By Proposition 1.4,  $\text{inj. dim}(H_{\mathfrak{p}}^{n-1}(R)) = 1$ . Let  $0 \rightarrow H_{\mathfrak{p}}^{n-1}(R) \rightarrow E^0 \rightarrow E^1 \rightarrow 0$  be the minimal injective resolution for  $H_{\mathfrak{p}}^{n-1}(R)$ . By Remark 2.2 (a), this is a sequence of  $\mathcal{D}$ -modules and  $\mathcal{D}$ -linear maps. We use Lemma 4.1 and [5, Theorem 2.4(d)] to deduce that  $E^1 \cong E_R(R/\mathfrak{m})^s$  where  $s$  is a finite positive integer and  $\mathfrak{m}$  is the maximal ideal of  $R$ . Thus  $E^1$  is a holonomic  $\mathcal{D}$ -module. It implies that  $E^0$  is a holonomic  $\mathcal{D}$ -module, see Remark 2.1(d). Note that  $\text{Ass}_R(H_{\mathfrak{p}}^{n-1}(R)) = \mathfrak{p}$ . Hence  $E_R(R/\mathfrak{p})$  is a submodule of  $E^0$  and therefore is a holonomic  $\mathcal{D}$ -module.  $\square$

**Lemma 4.4.** *Let  $(R, \mathfrak{m})$  be regular local ring of dimension  $n$  containing a field of characteristic zero. Let  $\mathfrak{p} \in \text{Spec}(R)$  such that  $\text{ht}_R(\mathfrak{p}) = n - 1$ . We denote the completion of  $R$  with respect to maximal ideal  $\mathfrak{m}$  by  $\hat{R}$ . Then  $E_R(R/\mathfrak{p}) \otimes_R \hat{R}$  has a structure of holonomic  $\mathcal{D}$ -module.*

*Proof.* Recall that  $E_R(R/\mathfrak{p}) \cong H_{\mathfrak{p}}^{n-1}(R)_{\mathfrak{p}}$ . Note that  $E_R(R/\mathfrak{p}) \otimes_R \hat{R} \cong H_{\mathfrak{p}}^{n-1}(R)_{\mathfrak{p}} \otimes_R \hat{R} \cong H_{\mathfrak{p}\hat{R}}^{n-1}(\hat{R})_{\mathfrak{p}}$ . By Remark 2.2(b),  $E_R(R/\mathfrak{p}) \otimes_R \hat{R}$  has a structure of  $\mathcal{D}(\hat{R}, k)$ -module where  $k$  is a field of characteristic zero which contained in  $\hat{R}$ . We simply say that  $E_R(R/\mathfrak{p}) \otimes_R \hat{R}$  is a  $\mathcal{D}$ -module. Keep in mind that  $\dim_{\hat{R}}(E_R(R/\mathfrak{p}) \otimes_R \hat{R}) = 1$  and for all  $\mathfrak{q} \in \text{Ass}_{\hat{R}}(E_R(R/\mathfrak{p}) \otimes_R \hat{R})$  we have  $\text{ht}_{\hat{R}}(\mathfrak{q}) \geq n - 1$ . Considering that  $\text{Ass}_{\hat{R}}(E_R(R/\mathfrak{p}) \otimes_R \hat{R})$  is finite and in view of Lemma 4.3 and [5, Theorem 2.4(d)], injective envelope of  $E_R(R/\mathfrak{p}) \otimes_R \hat{R}$  is a holonomic  $\mathcal{D}$ -module. It implies that  $E_R(R/\mathfrak{p}) \otimes_R \hat{R}$  is a holonomic  $\mathcal{D}$ -module.  $\square$

Now we are ready to prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2:

It is enough to prove the Proposition for  $2 \leq \dim(H_1^i(R)) \leq 3$ .

We denote the completion of  $R$  with respect to the maximal ideal  $\mathfrak{m}$  by  $\hat{R}$ . By the Cohen structure theorem  $\hat{R} = k[[x_1, \dots, x_n]]$  where  $k$  is a field of characteristic zero.

First we consider the case  $\dim_R(H_1^i(R)) = 2$ . Suppose on the contrary that  $\text{inj. dim}_R(H_1^i(R)) = 0$ . Thus  $H_1^i(R) \cong \bigoplus_{\mathfrak{p} \in \text{Ass}_R(H_1^i(R))} E_R(R/\mathfrak{p})^{\mu_0(\mathfrak{p}, H_1^i(R))}$ . There exists  $\mathfrak{q} \in \text{Ass}_R(H_1^i(R))$  such that  $\text{ht}_R(\mathfrak{q}) = n - 2$ . Thus the natural map  $E_R(R/\mathfrak{q}) \otimes_R \hat{R} \rightarrow H_{1\hat{R}}^i(\hat{R})$  is injective. It is contradiction because by Lemma 3.2  $E_R(R/\mathfrak{q}) \otimes_R \hat{R}$  cannot be holonomic.

Now let  $\dim_R(H_1^i(R)) = 3$ . Pick  $\mathfrak{p} \in \text{Supp}_R(H_1^i(R))$  such that  $\text{ht}_R(\mathfrak{p}) = n - 1$  and  $\dim_{R_{\mathfrak{p}}}(H_1^i(R)_{\mathfrak{p}}) = 2$ . Therefore  $1 \leq \text{inj. dim}_{R_{\mathfrak{p}}}(H_1^i(R)_{\mathfrak{p}}) \leq 2$ . If  $\text{inj. dim}_{R_{\mathfrak{p}}}(H_1^i(R)_{\mathfrak{p}}) = 2$  we are done. So suppose  $\text{inj. dim}_{R_{\mathfrak{p}}}(H_1^i(R)_{\mathfrak{p}}) = 1$ . Let  $0 \rightarrow H_1^i(R)_{\mathfrak{p}} \rightarrow E^0 \rightarrow E^1 \rightarrow 0$  be the



minimal injective resolution for  $H_1^i(R)_\mathfrak{p}$  as  $R_\mathfrak{p}$ -module. We claim that there are infinitely many prime ideals  $\mathfrak{q} \subseteq \mathfrak{p}$  such that  $\text{ht}_R(\mathfrak{q}) = n - 2$  and  $\mu_1(\mathfrak{q}, H_1^i(R)) > 0$ .

Proof of the claim: Suppose on the contrary that

$$E^1 \cong E_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})^t \oplus E_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{q}_1R_\mathfrak{p})^{t_1} \oplus \dots \oplus E_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{q}_sR_\mathfrak{p})^{t_s}$$

such that  $\text{ht}_R(\mathfrak{q}_i) = n - 2$  for all  $1 \leq i \leq s$ . Note that there is no prime  $\mathfrak{q} \subset \mathfrak{p}$  such that  $\mu_1(\mathfrak{q}, H_1^i(R)) > 0$  and  $\text{ht}(\mathfrak{q}) = n - 3$  because if there is one then  $1 = \text{inj. dim}_{R_\mathfrak{q}}(H_1^i(R)_\mathfrak{q}) > \dim_{R_\mathfrak{q}}(H_1^i(R)_\mathfrak{q}) = 0$  and it is impossible. We denote the completion of  $R_\mathfrak{p}$  with respect to the maximal ideal  $\mathfrak{p}R_\mathfrak{p}$  by  $\hat{R}_\mathfrak{p}$ . By the Cohen structure Theorem  $\hat{R}_\mathfrak{p} \cong k'[[y_1, \dots, y_{n-1}]]$  where  $k'$  is a field of characteristic zero. In view of Remark 2.2 (c), the exact sequence

$$0 \rightarrow H_1^i(R)_\mathfrak{p} \otimes_{R_\mathfrak{p}} \hat{R}_\mathfrak{p} \rightarrow E^0 \otimes_{R_\mathfrak{p}} \hat{R}_\mathfrak{p} \rightarrow E^1 \otimes_{R_\mathfrak{p}} \hat{R}_\mathfrak{p} \rightarrow 0$$

is a sequence of  $\mathcal{D}$ -modules and  $\mathcal{D}$ -linear maps. By Lemma 4.4,  $E^1 \otimes_{R_\mathfrak{p}} \hat{R}_\mathfrak{p}$  is a holonomic  $\mathcal{D}$ -module. Keep in mind that  $H_1^i(R)_\mathfrak{p} \otimes_{R_\mathfrak{p}} \hat{R}_\mathfrak{p} \cong H_{1\hat{R}_\mathfrak{p}}^i(\hat{R}_\mathfrak{p})$ . Thus  $H_1^i(R)_\mathfrak{p} \otimes_{R_\mathfrak{p}} \hat{R}_\mathfrak{p}$  is holonomic. So we deduce that  $E^0 \otimes_{R_\mathfrak{p}} \hat{R}_\mathfrak{p}$  is holonomic. There exists  $\mathfrak{q}R_\mathfrak{p} \in \text{Ass}_{R_\mathfrak{p}}(H_1^i(R)_\mathfrak{p})$  such that  $\text{ht}_{R_\mathfrak{p}}(\mathfrak{q}R_\mathfrak{p}) = n - 2$ . Therefore  $E_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}) \otimes_{R_\mathfrak{p}} \hat{R}_\mathfrak{p}$  is holonomic and this contradicts with Lemma 3.2. This proves the claim.

Now we suppose on the contrary that  $\text{inj. dim}_R(H_1^i(R)) = 1$ . By the claim there are infinitely many prime ideals  $\mathfrak{q} \in \text{Supp}_R(H_1^i(R))$  such that  $\text{ht}_R(\mathfrak{q}) = n - 2$  and  $\mu_1(\mathfrak{q}, H_1^i(R)) > 0$ . Pick one of these primes such that  $\mathfrak{q} \notin \text{Ass}_R(H_1^i(R))$ . Note that  $\text{Ass}_R(H_1^i(R))$  is finite. Let

$$E^\bullet : 0 \rightarrow H_1^i(R) \rightarrow E^0 \rightarrow E^1 \rightarrow 0$$

be the minimal injective resolution of  $H_1^i(R)$ . Then we have the following complex of  $R$ -modules:

$$0 \longrightarrow \Gamma_\mathfrak{q}(H_1^i(R)) \longrightarrow \Gamma_\mathfrak{q}(E^0) \longrightarrow \Gamma_\mathfrak{q}(E^1) \longrightarrow 0.$$

By assumption  $E_R(R/\mathfrak{q})$  is a submodule of  $\Gamma_\mathfrak{q}(E^1)$ . By tensoring the latter complex in  $\hat{R}$ , we get the following complex of  $\hat{R}$ -modules :

$$E^{\bullet\bullet} : 0 \longrightarrow \Gamma_\mathfrak{q}(H_1^i(R)) \otimes_R \hat{R} \longrightarrow \Gamma_\mathfrak{q}(E^0) \otimes_R \hat{R} \xrightarrow{\alpha} \Gamma_\mathfrak{q}(E^1) \otimes_R \hat{R} \xrightarrow{\beta} 0.$$

It is clear that  $E^{\bullet\bullet} \cong \Gamma_{\mathfrak{q}\hat{R}}(E^\bullet \otimes_R \hat{R})$ . By Remark 2.2(c),  $E^\bullet \otimes_R \hat{R}$  is a sequence of  $\mathcal{D}$ -modules and  $\mathcal{D}$ -linear maps. Thus  $E^{\bullet\bullet}$  is a complex of  $\mathcal{D}$ -modules and  $\mathcal{D}$ -linear maps. By assumption and Lemma 4.4,  $\Gamma_\mathfrak{q}(E^0) \otimes_R \hat{R}$  is a holonomic  $\mathcal{D}$ -module. So  $\text{im } \alpha$  is holonomic. On the other hand,  $\ker \beta / \text{im } \alpha \cong H_\mathfrak{q}^1(H_1^i(R)) \otimes_R \hat{R} \cong H_{\mathfrak{q}\hat{R}}^1(H_{1\hat{R}}^i(\hat{R}))$  is a holonomic  $\mathcal{D}$ -module. Thus  $\Gamma_\mathfrak{q}(E^1) \otimes_R \hat{R}$  is holonomic and it implies that  $E_R(R/\mathfrak{q}) \otimes_R \hat{R}$  is holonomic which is contradiction in view of Lemma 3.2.

Proof of Theorem 1.3:

Pick  $\mathfrak{p} \in \text{Supp}_R(H_1^i(R))$  such that  $\text{ht}(\mathfrak{p}) = n - 1$  and  $\dim_{R_\mathfrak{p}}(H_1^i(R))_\mathfrak{p} = 3$ . By Lemma 1.2,  $2 \leq \text{inj. dim}_{R_\mathfrak{p}}(H_1^i(R))_\mathfrak{p} \leq 3$ . If  $\text{inj. dim}_{R_\mathfrak{p}}(H_1^i(R))_\mathfrak{p} = 3$  we are done. Hence let

$\text{inj. dim}_{R_{\mathfrak{p}}}(H_1^i(R))_{\mathfrak{p}} = 2$ . Let

$$0 \rightarrow H_1^i(R)_{\mathfrak{p}} \rightarrow E^0 \rightarrow E^1 \rightarrow E^2 \rightarrow 0$$

be the minimal injective resolution of  $H_1^i(R)_{\mathfrak{p}}$  as  $R_{\mathfrak{p}}$  module.

We claim that, there exists  $\mathfrak{q} \in \text{Supp}_R(H_1^i(R))$  such that  $\mathfrak{q} \subset \mathfrak{p}$  and  $\text{ht}_R(\mathfrak{q}) = n - 2$  and  $\mu_2(\mathfrak{q}, H_1^i(R)) > 0$ . Therefore, as a consequence of Lemma 3.4,  $3 \leq \text{inj. dim}(H_1^i(R))$ .

Proof of the claim:

Suppose on the contrary that  $E^2 \cong E_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}})^t$ . Note that there is no prime  $\mathfrak{q} \subset \mathfrak{p}$  such that  $\mu_2(\mathfrak{q}, H_1^i(R)) > 0$  and  $\text{ht}_R(\mathfrak{q}) = n - 3$  or  $\text{ht}_R(\mathfrak{q}) = n - 4$ . Because if there is one then  $\text{inj. dim}_{R_{\mathfrak{q}}}(H_1^i(R)_{\mathfrak{q}}) > \dim_{R_{\mathfrak{q}}}(H_1^i(R)_{\mathfrak{q}})$  and it is impossible. Pick  $\mathfrak{q} \in \text{Supp}_R(H_1^i(R))$  which contained in  $\mathfrak{p}$  such that  $\text{ht}_R(\mathfrak{q}) = n - 2$  and  $\dim_{R_{\mathfrak{q}}}(H_1^i(R)_{\mathfrak{q}}) = 2$ . So

$$0 \rightarrow H_1^i(R)_{\mathfrak{q}} \rightarrow (E^0)_{\mathfrak{q}} \rightarrow (E^1)_{\mathfrak{q}} \rightarrow 0$$

is the minimal injective resolution of  $H_1^i(R)_{\mathfrak{q}}$  as  $R_{\mathfrak{q}}$  module. By the claim of the proof of Lemma 1.2, we can find  $\mathfrak{q}' \in \text{Supp}_R(H_1^i(R))$  which contained in  $\mathfrak{q}$  and  $\text{ht}_R(\mathfrak{q}') = n - 3$  such that  $\mu_1(\mathfrak{q}', H_1^i(R)) > 0$  and  $\mathfrak{q}' \notin \text{Ass}_R(H_1^i(R))$ .

Hence we have the complex

$$0 \rightarrow \Gamma_{\mathfrak{q}'}(H_1^i(R)_{\mathfrak{p}}) \rightarrow \Gamma_{\mathfrak{q}'}(E^0) \rightarrow \Gamma_{\mathfrak{q}'}(E^1) \rightarrow \Gamma_{\mathfrak{q}'}(E^2) \rightarrow 0$$

of  $R_{\mathfrak{p}}$  modules. By tensoring the latter complex in  $\hat{R}_{\mathfrak{p}}$ , completion of  $R_{\mathfrak{p}}$  with respect to the maximal ideal  $\mathfrak{p}R_{\mathfrak{p}}$ , we get the following complex which by using Remark 2.2(c) is a complex of  $\mathcal{D}$ -modules and  $\mathcal{D}$ -linear maps:

$$0 \longrightarrow \Gamma_{\mathfrak{q}'}(H_1^i(R)_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} \hat{R}_{\mathfrak{p}} \longrightarrow \Gamma_{\mathfrak{q}'}(E^0) \otimes_{R_{\mathfrak{p}}} \hat{R}_{\mathfrak{p}} \xrightarrow{\alpha} \Gamma_{\mathfrak{q}'}(E^1) \otimes_{R_{\mathfrak{p}}} \hat{R}_{\mathfrak{p}} \xrightarrow{\beta} \Gamma_{\mathfrak{q}'}(E^2) \otimes_{R_{\mathfrak{p}}} \hat{R}_{\mathfrak{p}} \longrightarrow 0.$$

By assumption  $\Gamma_{\mathfrak{q}'}(E^0) \otimes_{R_{\mathfrak{p}}} \hat{R}_{\mathfrak{p}}$  and  $\Gamma_{\mathfrak{q}'}(E^2) \otimes_{R_{\mathfrak{p}}} \hat{R}_{\mathfrak{p}}$  are holonomic  $\mathcal{D}$ -modules. So  $\text{im } \alpha$  and  $\text{im } \beta$  are holonomic. On the other hand  $\ker(\beta)/\text{im}(\alpha) \cong H_{\mathfrak{q}'R_{\mathfrak{p}}}^1(H_{1R_{\mathfrak{p}}}(R_{\mathfrak{p}})) \otimes_{R_{\mathfrak{p}}} \hat{R}_{\mathfrak{p}}$  is holonomic. So  $\ker(\beta)$  is holonomic and therefore  $\Gamma_{\mathfrak{q}'}(E^1) \otimes_{R_{\mathfrak{p}}} \hat{R}_{\mathfrak{p}}$  is holonomic. It follows that  $E_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{q}'R_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} \hat{R}_{\mathfrak{p}}$  is holonomic. Now we reach to a contradiction by Lemma 3.2.

**Corollary 4.5.** *The following hold.*

- a) *Let  $k$  be a field of characteristic zero and  $R = k[[x_1, \dots, x_5]]$ . Let  $I$  be an ideal of  $R$ . Then  $\dim_R(H_1^i(R)) - 1 \leq \text{inj. dim}_R(H_1^i(R))$ .*
- b) *Let  $(R, \mathfrak{m})$  be a regular local ring of dimension less than 5, which contains a field of characteristic zero. Let  $I$  be an ideal of  $R$ . Then  $\dim_R(H_1^i(R)) - 1 \leq \text{inj. dim}_R(H_1^i(R))$ .*

*Proof.* It is obvious by using Lemmas 1.2 and 1.3. Note that in (a),  $\dim_R(H_1^i(R)) \leq 4$  and in (b)  $\dim_R(H_1^i(R)) \leq 3$ .  $\square$

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